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On the Maximum Weighted Sum-Rate of MIMO Gaussian Broadcast Channels

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***Abstract*—In this paper, we investigate the maximum weighted sum-rate problem (MWSR) of MIMO Gaussian broadcast channels (MIMO-BC). We propose an efficient algorithm that employs conjugate gradient projections (CGP) to solve the MWSR problem. The proposed CGP offers provable convergence. By deflecting gradient direction to its Hessian conjugate, CGP enjoys a superlinear convergence rate. Also, CGP has a modest memory requirement. It only needs the solution information from the previous step. More importantly, CGP is able to solve the MWSR problem with arbitrary number of antennas on both sides of a MIMO-BC.**

I. Introduction

The capacity region of multiple-input multiple-output broadcast channels (MIMO-BC) has received great attention in recent years. MIMO-BC belongs to the class of nondegraded broadcast channels, for which the capacity region is a well-known hard problem [1]. Recently, Weigarten *et al.* [2] proved that "dirty paper coding" (DPC) achieves the entire capacity region of MIMO-BC. Moreover, by uplink-downlink duality [3], the nonconvex MIMO-BC capacity region (with respect to the downlink input covariance matrices) can be transformed to its dual MIMO multiple access channel (MIMO-MAC) capacity region with a sum power constraint. Since the capacity region of the dual MIMO-MAC is convex with respect to the uplink input covariance matrices, efficient optimization for MIMO-BC becomes possible.

In this paper, we investigate the maximum weighted sum-rate problem (MWSR) of MIMO-BC. Important applications of MWSR arise from cross-layer optimization for MIMO-based ad hoc networks [4] and stabilizing the transmission buffers to guarantee fairness for MIMO-based cellular downlinks [5]. The MWSR problem of MIMO-BC is the general case of the maximum sum-rate problem (MSR), which has been solved by a number of algorithms such as the minimax method (MM) [6], the gradient method (GD) [7], the Lagrangian dual decomposition (LDD) method [8], and the iterative water-filling methods (IWF) [9]. However, IWF, MM, and LDD cannot be readily applied to solve MWSR. In [5], Kobayashi *et al.* extended IWF to solve the MWSR problem and proposed some modifications to IWF (M-IWF) to handle scalability issue. However, their algorithm is only valid for the case where each receiver is equipped with single antenna. For general scenarios where receivers are equipped with multiple antennas, only GD is readily applicable. However, GD does not fully take advantage of the gradient information and may not converge under some circumstances. The limitations of these existing algorithms motivate us to design an efficient, robust, and scalable algorithm to solve the MWSR problem of large MIMO-BC systems with *arbitrary* number of antennas.

Our main contribution in this paper is that we design an efficient algorithm to solve the MWSR problem based on conjugate gradient projection (CGP) approach. Our algorithm is inspired by [10], where Ye *et al.* used a gradient projection method to find a local optimum of the maximum sum-rate for Gaussian MIMO Interference Channels (MIMO-IC). However, unlike [10], we propose to use *conjugate* gradient directions instead of gradient directions to reduce the "zigzagging" phenomenon so as to speed up convergence. Also, since the MWSR problem of MIMO-BC can be transformed into an equivalent convex problem, our CGP method can determine the global optimum of MIMO-BC. For the semidefinite cone projection subproblem, we develop a rigorous algorithm based on Lagrangian duality. Our proposed CGP has the following attractive features.

- CGP offers provable convergence.
- Unlike M-IWF, which is only valid for cases where each receiver has a single antenna, CGP can handle arbitrary number of antennas on both sides of a MIMO-BC.
- CGP enjoys a superlinear convergence rate. Also, per iteration complexity of CGP is $O(K)$, where K is the number of users.
- CGP has a modest memory requirement: it only needs the solution information from the previous step.

The remainder of this paper is organized as follows. In Section II, we present the network model and the problem formulation. Section III introduces the key components in of CGP, including the computation of conjugate gradients and solving projection subproblem. In Section IV, we analyze the complexity of CGP and present numerical results. Section V concludes this paper.

II. System Model and Problem Formulation

We begin with introducing notation. We use boldface to denote matrices and vectors. For a complex-valued matrix $\mathbf{A}$, $\mathbf{A}^\dagger$ denote the conjugate transpose of $\mathbf{A}$ and $\mathrm{Tr}\{\mathbf{A}\}$ denotes the trace of $\mathbf{A}$. We let $\mathbf{I}$ denote an identity matrix with dimension determined from the context. $\mathbf{A} \succeq 0$ represents that $\mathbf{A}$ is Hermitian and positive semidefinite (PSD). $\mathrm{Diag}\{\mathbf{A}_1 \ldots \mathbf{A}_n\}$

denotes the block diagonal matrix with matrices $\mathbf{A}_1, \ldots, \mathbf{A}_n$ on its main diagonal.

Suppose that a MIMO Gaussian broadcast channel has K users, the transmitter has n_t antennas, and each of the K users is equipped with n_r antennas. The channel matrix for user i is denoted as $\mathbf{H}_i \in \mathbb{C}^{n_r \times n_t}$. It has been shown in [2] that the capacity region of a MIMO-BC is equal to the dirty-paper coding (DPC) rate region. Suppose that users $1, \ldots, K$ are encoded sequentially, then the DPC rate of user i can be computed as [3]

$$C_i^{\mathrm{DPC}}(\mathbf{S}) = \log \frac{\left|\mathbf{I} + \mathbf{H}_i \left(\sum_{j=i}^{K} \mathbf{S}_j\right) \mathbf{H}_i^{\dagger}\right|}{\left|\mathbf{I} + \mathbf{H}_i \left(\sum_{j=i+1}^{K} \mathbf{S}_j\right) \mathbf{H}_i^{\dagger}\right|}, \qquad (1)$$

where $\mathbf{S}_i \in \mathbb{C}^{n_t \times n_t}$, $i = 1, \ldots, K$, are the *downlink* input covariance matrices and $\mathbf{S} \triangleq \{\mathbf{S}_1, \ldots \mathbf{S}_K\}$ denotes the collection of all the downlink covariance matrices. The MWSR problem can be written as follows:

$$\begin{array}{ll} \textsf{Maximize} & \sum_{i=1}^{K} w_i C_i^{\mathrm{DPC}}(\mathbf{S}) \\ \textsf{subject to} & \mathbf{S}_i \succeq 0, \quad i = 1, \ldots, K \\ & \sum_{i=1}^{K} \mathrm{Tr}(\mathbf{S}_i) \leq P, \end{array} \qquad (2)$$

where w_i is the weight assigned to user i, P represents the maximum transmit power. It is evident that (2) is a nonconvex optimization problem since the DPC rate equation in (1) is a nonconvex function of the input covariance matrices $\mathbf{S}_1, \ldots, \mathbf{S}_K$. However, from the uplink-downlink duality theorem [3], we know that the rates achievable in a MIMO-BC are also achievable in its dual MIMO-MAC. That is, given a feasible $\mathbf{S}$, there exists a set of feasible *uplink* input covariance matrices for its dual MIMO-MAC, denoted by $\mathbf{Q}$, such that $C_i^{\mathrm{MAC}}(\mathbf{Q}) = C_i^{\mathrm{DPC}}(\mathbf{S})$. Thus, (2) is equivalent to the following MWSR problem of the dual MIMO-MAC with a sum power constraint:

$$\begin{array}{ll} \textsf{Maximize} & \sum_{i=1}^{K} w_i C_i^{\mathrm{MAC}}(\mathbf{Q}) \\ \textsf{subject to} & \mathbf{Q}_i \succeq 0, \quad i = 1, \ldots, K \\ & C_i^{\mathrm{MAC}}(\mathbf{Q}) \in \mathcal{C}_{\mathrm{MAC}}(P, \mathbf{H}^{\dagger}),\ i = 1, \ldots, K \\ & \sum_{i=1}^{K} \mathrm{Tr}(\mathbf{Q}_i) \leq P, \end{array} \qquad (3)$$

where $\mathbf{Q}_i \in \mathbb{C}^{n_r \times n_r}$, $i = 1, \ldots, K$, are the uplink input covariance matrices, $\mathbf{Q} \triangleq \{\mathbf{Q}_1, \ldots \mathbf{Q}_K\}$ represents the collection of all the uplink covariance matrices, and $\mathcal{C}_{\mathrm{MAC}}(P, \mathbf{H}^{\dagger})$ represents the capacity region of the dual MIMO-MAC, and can be determined by

$$\mathcal{C}_{\mathrm{MAC}}(P, \mathbf{H}^{\dagger}) = \\ Conv \left\{ (C_1, \ldots, C_K) \left| \begin{array}{l} \sum_{i \in \mathcal{S}} C_i(\mathbf{Q}) \leq \\ \log \left|\mathbf{I} + \sum_{i \in \mathcal{S}} \mathbf{H}_i^{\dagger} \mathbf{Q}_i \mathbf{H}_i\right|, \\ \forall \mathcal{S} \subseteq \{1, \ldots, K\}, \\ \sum_{i=1}^{K} \mathrm{Tr}(\mathbf{Q}_i) \leq P, \\ \mathbf{Q}_i \succeq 0,\ \forall i. \end{array} \right. \right\}, \qquad (4)$$

where $Conv(\cdot)$ represents the convex hull operation. When the dual MIMO-MAC is Gaussian, the convex hull operation can be dropped [1].

The capacity region of a MIMO-MAC can be achieved by successive decoding [1]. However, in order to determine the capacity region of a MIMO-MAC, $K!$ possible successive decoding orders may need to be enumerated, which makes the problem intractable if the number of users is large. We give the following result, which shows that such an enumeration can indeed be avoided.

Theorem 1. *The MWSR problem in (3) can be solved by the following equivalent optimization problem:*

$$\begin{array}{ll} \textsf{Maximize} & \sum_{i=1}^{K} (w_{\pi(i)} - w_{\pi(i-1)}) \times \\ & \log \left|\mathbf{I} + \sum_{j=i}^{K} \mathbf{H}_{\pi(j)}^{\dagger} \mathbf{Q}_{\pi(j)} \mathbf{H}_{\pi(j)}\right| \\ \textsf{subject to} & \sum_{i=1}^{K} \mathrm{Tr}(\mathbf{Q}_i) \leq P_{\max} \\ & \mathbf{Q}_i \succeq 0,\ i = 1, \ldots, K, \end{array} \qquad (5)$$

where $w_{\pi(0)} \triangleq 0$, π is a permutation of the set $\{1, \ldots, K\}$ such that $w_{\pi(1)} \leq \ldots \leq w_{\pi(K)}$ (i.e., $\pi(i) = j$ represents the i^{th} position in permutation π is user j).

Proof. Since the objective function is monotonically increasing, the optimal solution of (3) must be achieved on the boundary of the capacity region. We assume that the weights are not identical (Otherwise, the MWSR is reduced to a scaled MSR problem, where the optimal solution is trivially achieved at any of the $K!$ corner points of the capacity region). As a result, the optimal solution is achieved in a subregion on the boundary of the capacity region under one of the $K!$ decoding order [1]. Furthermore, since such a subregion consists of all corner points with the same decoding order under all feasible power allocations, the optimal solution must be achieved at a corner point corresponding to some decoding order and some power allocation.

Suppose that $\pi(\cdot)$ is the optimal decoding order. It is easy to see that, for the MWSR problem, the objective gradient at every point on the boundary of the capacity region is

$$\left[\begin{array}{cccc} w_{\pi(1)} & w_{\pi(2)} & \cdots & w_{\pi(K)} \end{array}\right]^T.$$

Since the objective function of (3) is linear and the capacity region is convex, (3) is a convex programming problem with non-empty feasible region, which means Slater conditions holds (c.f. [11]). As a result, KKT condition is necessary and sufficient for optimality.

For simplicity, we drop the superscript "MAC" and simply refer the rates in the dual MIMO-MAC as C_i. Note that in successive decoding with decoding order $\pi(\cdot)$, the active rate vector constraints at a corner point corresponding to $\pi(\cdot)$ are

$$\begin{array}{l} C_{\pi(K)} \leq \log \left|\mathbf{I} + \mathbf{H}_{\pi(K)}^{\dagger} \mathbf{Q}_{\pi(K)}^{*} \mathbf{H}_{\pi(K)}\right|, \\ C_{\pi(K-1)} + C_{\pi(K)} \leq \\ \qquad \log \left|\mathbf{I} + \sum_{i=K-1}^{K} \mathbf{H}_{\pi(i)}^{\dagger} \mathbf{Q}_{\pi(i)}^{*} \mathbf{H}_{\pi(i)}\right|, \\ \qquad \vdots \\ C_{\pi(1)} + \cdots + C_{\pi(K-1)} + C_{\pi(K)} \leq \\ \qquad \log \left|\mathbf{I} + \sum_{i=1}^{K} \mathbf{H}_{\pi(i)}^{\dagger} \mathbf{Q}_{\pi(i)}^{*} \mathbf{H}_{\pi(i)}\right|, \end{array} \qquad (6)$$

where $\mathbf{Q}^*_{\pi(i)}$, $i = 1, 2, \ldots, K$ represent the optimal input covariance matrices that achieve the maximum weighted sum-rate. Thus, by KKT condition, we have

$$\begin{bmatrix} w_{\pi(1)} \\ \vdots \\ w_{\pi(K-1)} \\ w_{\pi(K)} \end{bmatrix} = u_1 \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix} + \cdots + u_K \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}, \quad (7)$$

where $u_i \geq 0, \forall i$. Solving for u_i in (7), we have

$$\begin{aligned} u_K &= w_{\pi(1)} \\ u_{K-i} &= w_{\pi(i+1)} - w_{\pi(i)}, \ i = 1, 2, \ldots, K-1. \end{aligned}$$

Since $u_i \geq 0$, it then follows that

$$w_{\pi(1)} \leq w_{\pi(2)} \leq \ldots \leq w_{\pi(K)}.$$

Since the constraints in (6) are tight at optimality, we have

$$C_{\pi(K)} = \log \left| \mathbf{I} + \mathbf{H}^\dagger_{\pi(K)} \mathbf{Q}^*_{\pi(K)} \mathbf{H}_{\pi(K)} \right|, \quad (8)$$

$$C_{\pi(i)} = \log \left| \mathbf{I} + \sum_{j=i}^{K} \mathbf{H}^\dagger_{\pi(K)} \mathbf{Q}^*_{\pi(K)} \mathbf{H}_{\pi(K)} \right| - \log \left| \mathbf{I} + \sum_{j=i+1}^{K} \mathbf{H}^\dagger_{\pi(K)} \mathbf{Q}^*_{\pi(K)} \mathbf{H}_{\pi(K)} \right|, \quad (9)$$

for $i = 1, 2, \ldots, K-1$. Summing up all $w_{\pi(i)} C_{\pi(i)}$ and after rearranging the terms, it can be readily verified that

$$\sum_{i=1}^{K} w_{\pi(i)} C_{\pi(i)} = \sum_{i=1}^{K} (w_{\pi(i)} - w_{\pi(i-1)}) \times \log \left| \mathbf{I} + \sum_{j=i}^{K} \mathbf{H}^\dagger_{\pi(j)} \mathbf{Q}_{\pi(j)} \mathbf{H}_{\pi(j)} \right|. \quad (10)$$

It then follows that the MWSR problem of the dual MIMO-MAC is equivalent to maximizing (10) with the sum power constraint, i.e., the optimization problem in (5). $\square$

One important observation from (5) is that, since $\log|\cdot|$ is a concave function for positive semidefinite matrices [1], (5) is a convex optimization problem with respect to the uplink input covariance matrices $\mathbf{Q}_{\pi(1)}, \ldots, \mathbf{Q}_{\pi(K)}$. However, although standard convex optimization tools can be used to solve (5), it is considerably more complex than a custom-designed method that exploits the special structure of the problem.

III. Solving MWSR Using Conjugate Gradient Projection

To solve (5), we propose an efficient algorithm based on conjugate gradient projection (CGP), which utilizes an important concept called Hessian conjugate to deflect the gradient direction. In doing so, we can achieve an asymptotic *super-linear* convergence rate [11], which is close to that of quasi-Newton methods, e.g., BFGS method. The convergence proof of CGP relies on proving the closedness of the algorithmic maps for finding conjugate gradient directions and performing projections, respectively. Due to space limitation, we refer readers to [11] for details. The CGP pseudo-code for solving (5) is shown in Algorithm 1.

Algorithm 1 CGP Method for Solving MWSR

Initialization:

Choose the initial conditions $\mathbf{Q}^{(0)} = [\mathbf{Q}_1^{(0)}, \mathbf{Q}_2^{(0)}, \ldots, \mathbf{Q}_K^{(0)}]^T$. Let $k = 0$.

Main Loop:

1. Calculate the gradients $\mathbf{G}_i^{(k)}$, $i = 1, 2, \ldots, K$ as follows:
$$\bar{\mathbf{G}}_{\pi(j)} = 2\mathbf{H}_{\pi(j)} \left[\sum_{i=1}^{j} \left(w_{\pi(i)} - w_{\pi(i-1)}\right) \times \left(\mathbf{I} + \sum_{k=i}^{K} \mathbf{H}^\dagger_{\pi(k)} \mathbf{Q}_{\pi(k)} \mathbf{H}_{\pi(k)} \right)^{-1} \right] \mathbf{H}^\dagger_{\pi(j)}.$$
2. Deflect the gradients using Fletcher and Reeves' choice of deflection:
$$\rho_k = \frac{\|\bar{\mathbf{G}}^{(k)}_{\pi(j)}\|^2}{\|\bar{\mathbf{G}}^{(k-1)}_{\pi(j)}\|^2}.$$
3. Choose an appropriate step size s_k. Let $\mathbf{Q}_i^{'(k)} = \mathbf{Q}_i^{(k)} + s_k \mathbf{G}_i^{(k)}$, for $i = 1, 2, \ldots, K$.
4. Let $\bar{\mathbf{Q}}^{(k)}$ be the projection of $\mathbf{Q}^{'(k)}$ onto $\Omega_+(P)$, where $\Omega_+(P) \triangleq \{\mathbf{Q}_i, \ i = 1, \ldots, K | \mathbf{Q}_i \succeq 0, \sum_{i=1}^{K} \mathrm{Tr}\{\mathbf{Q}_i\} \leq P\}$.
5. Choose appropriate step size α_k. Let $\mathbf{Q}_i^{(k+1)} = \mathbf{Q}_i^{(k)} + \alpha_k (\bar{\mathbf{Q}}_i^{(k)} - \mathbf{Q}_i^{(k)})$, $i = 1, 2, \ldots, K$.
6. $k = k+1$. If the maximum absolute value of the elements in $\mathbf{Q}_i^{(k)} - \mathbf{Q}_i^{(k-1)} < \epsilon$, for $i = 1, 2, \ldots, L$, then stop; else go to step 1.

Due to the complexity of the objective function in (5), we adopt "Armijo Rule" inexact line search to avoid excessive objective function evaluations, while still enjoying provable convergence [11]. We now consider two major components in the CGP framework: 1) how to compute the conjugate gradient direction $\mathbf{G}_i$; and 2) how to project $\mathbf{Q}^{'(k)}$ onto the set $\Omega_+(P) \triangleq \{\mathbf{Q}_i, \ i = 1, \ldots, K : \mathbf{Q}_i \succeq 0, \sum_{i=1}^{K} \mathrm{Tr}\{\mathbf{Q}_i\} \leq P\}$.

A. Computing the Conjugate Gradients

For convenience, we denote the objective function of (5) as $J(\mathbf{Q})$. To compute the gradient $\bar{\mathbf{G}}_{\pi(j)} \triangleq \nabla_{\mathbf{Q}_{\pi(j)}} J(\mathbf{Q})$, the first step is to compute the partial derivative of $J(\mathbf{Q})$ with respect to $\mathbf{Q}_{\pi(j)}$. The computation of partial derivatives of $J(\mathbf{Q})$ relies on the following equation from matrix differential calculus $\frac{\partial \ln|\mathbf{A}+\mathbf{BXC}|}{\partial \mathbf{X}} = \left[\mathbf{C}(\mathbf{A}+\mathbf{BXC})^{-1}\mathbf{B}\right]^T$ [10], [12]. First of all, we can compute the partial derivative of the i^{th} term in the summation of $J(\mathbf{Q})$ with respect to $\mathbf{Q}_{\pi(j)}$, $j \geq i$, as follows:

$$\frac{\partial}{\partial \mathbf{Q}_{\pi(j)}} \left((w_{\pi(i)} - w_{\pi(i-1)}) \log \left| \mathbf{I} + \sum_{k=i}^{K} \mathbf{H}^\dagger_{\pi(k)} \mathbf{Q}_{\pi(k)} \mathbf{H}_{\pi(k)} \right| \right) = \left(w_{\pi(i)} - w_{\pi(i-1)}\right) \times \left[\mathbf{H}_{\pi(j)} \left(\mathbf{I} + \sum_{k=i}^{K} \mathbf{H}^\dagger_{\pi(k)} \mathbf{Q}_{\pi(k)} \mathbf{H}_{\pi(k)} \right)^{-1} \mathbf{H}^\dagger_{\pi(j)} \right]^T.$$

Note that for gradient $\bar{\mathbf{G}}_{\pi(j)}$, only the first j terms in $J(\mathbf{Q})$ involve $\mathbf{Q}_{\pi(j)}$. From the definition $\nabla_z f(z) = 2(\partial f(z)/\partial z)^*$

[13], we obtain

$$\bar{\mathbf{G}}_{\pi(j)} = 2\mathbf{H}_{\pi(j)} \left[\sum_{i=1}^{j} \left(w_{\pi(i)} - w_{\pi(i-1)} \right) \times \left(\mathbf{I} + \sum_{k=i}^{K} \mathbf{H}^{\dagger}_{\pi(k)} \mathbf{Q}_{\pi(k)} \mathbf{H}_{\pi(k)} \right)^{-1} \right] \mathbf{H}^{\dagger}_{\pi(j)}. \quad (11)$$

Although the (11) is seemingly quite complex, we can in fact exploit the special summation structure to reduce its computational complexity when implementing CGP. Note that the most computationally heavy part in the expression of $\bar{\mathbf{G}}_{\pi(j)}$ is the summation of the terms in the form of $\mathbf{H}^{\dagger}_{\pi(k)} \mathbf{Q}_{\pi(k)} \mathbf{H}_{\pi(k)}$. Under direct computation, we will have $j(2K+1-j)/2$ times of such additions for these terms. Fortunately, most of these terms in the summation occur repeatedly when j varies. Therefore, we can store a running sum in the form of $\mathbf{I} + \sum_{k=i}^{K} \mathbf{H}^{\dagger}_{\pi(k)} \mathbf{Q}_{\pi(k)} \mathbf{H}_{\pi(k)}$. Then, starting out from $j = K$ and reducing j by one subsequently, we only need to compute such addition once in each iteration.

The conjugate gradient direction in the k^{th} iteration can be computed as $\mathbf{G}^{(k)}_{\pi(j)} = \bar{\mathbf{G}}^{(k)}_{\pi(j)} + \rho_k \mathbf{G}^{(k-1)}_{\pi(j)}$. We adopt the Fletcher and Reeves' choice of deflection [11], which can be computed as $\rho_k = \|\bar{\mathbf{G}}^{(k)}_{\pi(j)}\|^2 / \|\bar{\mathbf{G}}^{(k-1)}_{\pi(j)}\|^2$. After such deflection, we obtained the so-called Hessian-conjugate of $\mathbf{G}^{(k-1)}_{\pi(j)}$. The benefit of using Hessian conjugate deflection is that we can reduce the "zigzagging" phenomenon encountered in the conventional gradient projection method, and achieve an asymptotic superlinear convergence rate [11]. Also, in CGP, we do not need to store a Hessian approximation matrix as in quasi-Newton methods, whose size is usually large.

B. Constrained Semidefinite Cone Projection

The goal of the projection subproblem in CGP is to find a projection on a constrained semidefinite cone for $\mathbf{Q}_i, \forall i$. Since $\mathbf{G}_{\pi(j)}$ is Hermitian, we have that $\mathbf{Q}'^{(k)}_{\pi(j)} = \mathbf{Q}^{(k)}_{\pi(j)} + s_k \mathbf{G}^{(k)}_{\pi(j)}$ is Hermitian as well. Then, the projection problem becomes how to simultaneously project K Hermitian matrices onto the set $\Omega_+(P_{\max}) \triangleq \{\mathbf{Q}_l : \sum_l \mathrm{Tr}\{\mathbf{Q}_l\} \leq P_{\max}, \mathbf{Q}_l \succeq 0, \; l = 1, \ldots, K\}$. We construct a block diagonal matrix $\mathbf{D} = \mathrm{Diag}\{\mathbf{Q}_{\pi(1)} \ldots \mathbf{Q}_{\pi(K)}\} \in \mathbb{C}^{(K \cdot n_r) \times (K \cdot n_r)}$. It is easy to recognize that $\mathbf{Q}_{\pi(j)} \in \Omega_+(P_{\max})$, $j = 1, \ldots, K$, if and only if $\mathrm{Tr}(\mathbf{D}) = \sum_{j=1}^{K} \mathrm{Tr}\left(\mathbf{Q}_{\pi(j)}\right) \leq P_{\max}$ and $\mathbf{D} \succeq 0$. In our projection, given a block diagonal matrix $\mathbf{D}_n$, we wish to find a matrix $\tilde{\mathbf{D}}_n \in \Omega_+(P_{\max})$ such that $\tilde{\mathbf{D}}_n$ minimizes $\|\tilde{\mathbf{D}}_n - \mathbf{D}_n\|_F$, where $\|\cdot\|_F$ denotes Frobenius norm, i.e., equivalently, we solve the following optimization problem.

$$\begin{array}{ll} \text{Minimize} & \frac{1}{2}\|\tilde{\mathbf{D}} - \mathbf{D}\|_F^2 \\ \text{subject to} & \mathrm{Tr}(\tilde{\mathbf{D}}) \leq P_{\max}, \; \tilde{\mathbf{D}} \succeq 0. \end{array} \quad (12)$$

Note that this problem is a convex minimization problem and we can solve this minimization problem by solving its Lagrangian dual. Associating Hermitian matrix $\mathbf{X}$ to the constraint $\tilde{\mathbf{D}} \succeq 0$ and μ to the constraint $\mathrm{Tr}(\tilde{\mathbf{D}}) \leq P_{\max}$, we can write the Lagrangian as $g(\mathbf{X}, \mu) = \min_{\tilde{\mathbf{D}}}\{(1/2)\|\tilde{\mathbf{D}} - \mathbf{D}\|_F^2 - \mathrm{Tr}(\mathbf{X}^{\dagger}\tilde{\mathbf{D}}) + \mu(\mathrm{Tr}(\tilde{\mathbf{D}}) - P_{\max})\}$. Since $g(\mathbf{X}, \mu)$ is an unconstrained convex quadratic minimization problem, we can compute the minimizer of the Lagrangian by simply setting its first derivative (with respect to $\tilde{\mathbf{D}}$) to zero, i.e., $(\tilde{\mathbf{D}} - \mathbf{D}) - \mathbf{X}^{\dagger} + \mu\mathbf{I} = 0$. Noting that $\mathbf{X}^{\dagger} = \mathbf{X}$, we have $\tilde{\mathbf{D}} = \mathbf{D} - \mu\mathbf{I} + \mathbf{X}$. Substituting $\tilde{\mathbf{D}}$ back into the Lagrangian and after some algebraic simplifications, we can rewrite the Lagrangian dual problem as

$$\begin{array}{ll} \text{Maximize} & -\frac{1}{2}\|\mathbf{D} - \mu\mathbf{I} + \mathbf{X}\|_F^2 - \mu P_{\max} + \frac{1}{2}\|\mathbf{D}\|^2 \\ \text{subject to} & \mathbf{X} \succeq 0, \mu \geq 0. \end{array} \quad (13)$$

In semidefinite programming, (13) is referred to as matrix nearness problems [14], [15]. Generic matrix nearness problems are hard to solve. Fortunately, thanks to the piecewise quadratic structure in (13), it is possible to solve (13) efficiently. Due to space limitation, we refer readers to [16] for more details.

IV. Performance and Complexity Comparisons

In this section, we compare CGP with other existing algorithms. Among these algorithms, the minimax method (MM) [6] is more complex than the others having the linear complexity and is not readily applicable for MWSR. The Lagrangian dual decomposition method (LDD) [8] consists of nested iterative loops in solving the Lagrangian dual in each iteration and therefore has a non-deterministic complexity per iteration. The iterative water-filling methods (IWF) in [9] do not scale well as the number of users increases because the most recently updated solution in each iteration only accounts for a fraction of $1/K$ in the effective channels' computation. Also, IWF cannot be directly used to solve MWSR. Kobayashi *et al.* proposed modifications of IWF (M-IWF) for solving MWSR. They also came up with a new averaging update scheme to address the scalability issue of IWF. However, as indicated by Kobayashi *et al.*, M-IWF can only handle the case when each receiver in a MIMO-BC has only one antenna.

The gradient method (GD) in [7] also uses the gradient information to guide the search of optimal solution. Let $\mathbf{v}_i$ and λ_i be the principal eigenvector (of unit norm) and principal eigenvalue for $\mathbf{Q}_i$, respectively, $i = 1, 2, \ldots, K$. Let $j^* = \arg\max(\lambda_1, \ldots, \lambda_K)$. The iterate of GD is updated as

$$\mathbf{Q}_i^{(k+1)} = \mathbf{Q}_i^{(k)} + t^* \mathbf{d}^{(k)},$$

where the moving direction is

$$\mathbf{d}^{(k)} = [-\mathbf{Q}_1^{(k)}, \; -\mathbf{Q}_2^{(k)}, \; \ldots \; -\mathbf{Q}_{j^*}^{(k)} + P\mathbf{v}_{j^*}\mathbf{v}_{j^*}^{\dagger}, \; \ldots \; -\mathbf{Q}_K^{(k)}]^T,$$

and the step-size t^* is determined by the following line search

$$t^* = \arg\max_{0 \leq t \leq 1} J((1-t)\mathbf{Q}_1^{(k)}, \ldots, (1-t)\mathbf{Q}_{j^*}^{(k)} + tP\mathbf{v}_{j^*}\mathbf{v}_{j^*}^{\dagger}, \ldots, (1-t)\mathbf{Q}_K^{(k)}),$$

where the interval $0 \leq t \leq 1$ ensures that the searching stays in the feasible region. The direction $\mathbf{d}^{(k)}$ is obtained by $\mathbf{v}_{j^*}\mathbf{v}_{j^*}^{\dagger}$ projected onto the hyperplane $\sum_{i=1}^{K} \mathrm{Tr}(\mathbf{Q}_i^{(k)}) = P$, which, although related to the gradient at $\mathbf{Q}^{(k)}$, is very different to our approach. In essence, $\mathbf{v}_{j^*}\mathbf{v}_{j^*}^{\dagger}$ is a rank-one update to $\mathbf{Q}_{j^*}^{(k)}$, as opposed to the full-rank update in CGP. In fact, GD is a variant of Zoutendijk method, for which the convergence is not guaranteed as the algorithmic map of Zoutendijk method is not closed [11]. Fig. 1 shows a equal-weighted 10-user equal-weighted MIMO-BC example where GD fails to converge to the optimal solution.

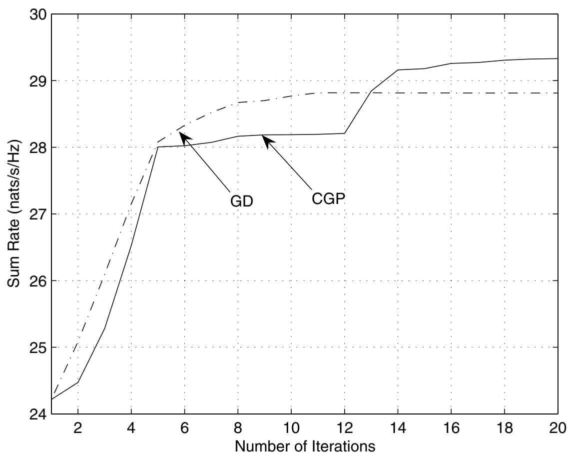

Fig. 1. A 10-user MIMO-BC channel with $n_t = n_r = 4$ where GD fails to converge to the optimal solution.

It can also be seen from Fig. 1 that CGP is very efficient. It only takes 20 iterations to reach to the global optimum. In fact, it can be shown that, by using conjugate gradient directions, CGP achieves an asymptotic superlinear convergence rate.

V. Conclusion

In this paper, we investigated the maximum weighted sum-rate problem (MWSR) of MIMO Gaussian broadcast channels (MIMO-BC). We proposed an efficient algorithm called conjugate gradient projections (CGP) to solve the MWSR problem. The proposed CGP has provable convergence. With appropriate deflections for gradients, CGP enjoys an asymptotic superlinear convergence rate. Also, CGP has a modest memory requirement, which only needs the solution information from the previous step. Another attractive feature of CGP is that it can handle arbitrary number of antennas on both sides of a MIMO-BC.

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